VAMO 2026 Team Round Problems and Solutions

Written by the VAMO Team

Problem 1

If $a * b = \frac{6}{b} + 3a$, then $2 * 3 = \frac{6}{3} + 3(2) = 8$. What is $4 * (1 * 2)$?

Solution:

$$1 * 2 = \frac{6}{2} + 3(1) = 6$$

$$4 * 6 = \frac{6}{6} + 3(4) = 1 + 12 = \boxed{13}$$

$\square$

Problem 2

What is the smallest positive integer that is divisible by 2, 3, and 4?

Solution: If a number is divisible by 4 then it is divisible by 2, so we only need to consider the least common multiple of 3 and 4, which is $\boxed{12}$. $\square$

Problem 3

Brian buys Pokemon™ card packs at a price of 10 packs for 50 dollars and sells them at a price of 8 packs for 90 dollars. How many packs will he have to sell to make a \$1000 profit?

Solution: Brian purchases packs at a price of \$5 per pack, and sells them for \$11.25 per pack. This is a profit of \$6.25 per pack. $1000 \div 6.25 = 160$ so Brian must purchase and sell $\boxed{160 \text{ packs}}$. $\square$

Problem 4

How many distinct permutations of the letters of the string "VAMO" do not have the two vowels next to each other?

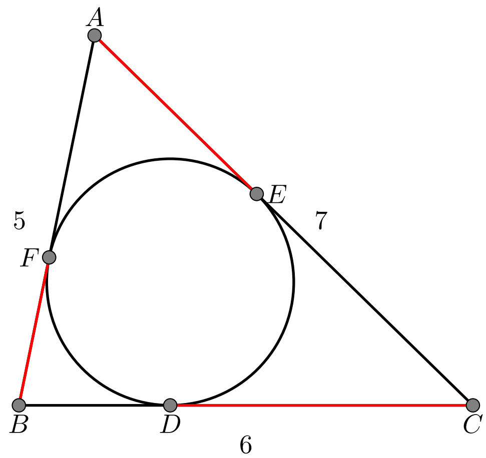
Solution: There are a total of $4! = 24$ possible permutations of the letters. The number with the vowels next to each other is $3! \cdot 2 = 12$, because we can think of the "AO" as one block ordered among the two other letters, and we multiply by 2 to account for the two ways to order "A" and "O" relative to each other. Thus, the number of valid words is

$$24 - 12 = \boxed{12}.$$

$\square$

Problem 5

Let triangle ABC have side lengths $AB = 5$, $BC = 6$, and $CA = 7$. Let circle Ω be inscribed inside $\triangle ABC$. Let it be tangent to BC, CA, and AB at points D, E, and F respectively. What is $AE + CD + BF$?



Solution: Note that $AE = AF$, $CD = CE$, and $BD = BF$. Since

$$AB + BC + CD = AE + CE + CD + BD + BF + AF = 2(AE + CD + BF) = 5 + 6 + 7 = 18$$

we obtain

$$AE + CD + BF = \boxed{9}$$

□

Problem 6

How many 3-digit positive integers are divisible by 9?

Solution: The smallest 3-digit multiple of 9 is $9 \cdot 12 = 108$, while the largest 3-digit multiple of 9 is $9 \cdot 111 = 999$. Therefore, there are a total of $111 - 12 + 1 = \boxed{100}$ 3-digit positive integers that are divisible by 9. □

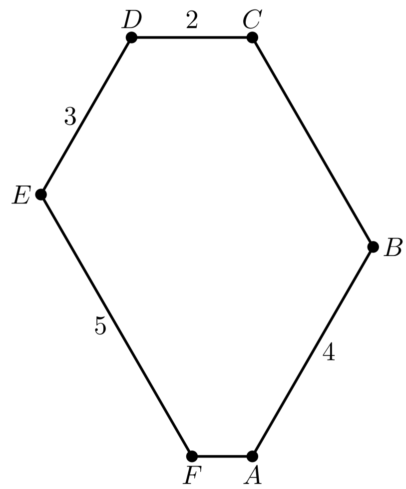
Problem 7

Eighty-eight less than the square of a number is equal to the square of the quantity that is 4 less than the number. What is the number?

Solution: We can express this statement as $x^2 - 88 = (x-4)^2$, which can be expanded to $x^2 - 88 = x^2 - 8x + 16$. Subtracting x^2 from both sides yields $8x = 104$, and dividing by 8 on both sides yields the answer $x = \boxed{13}$ □

Problem 8

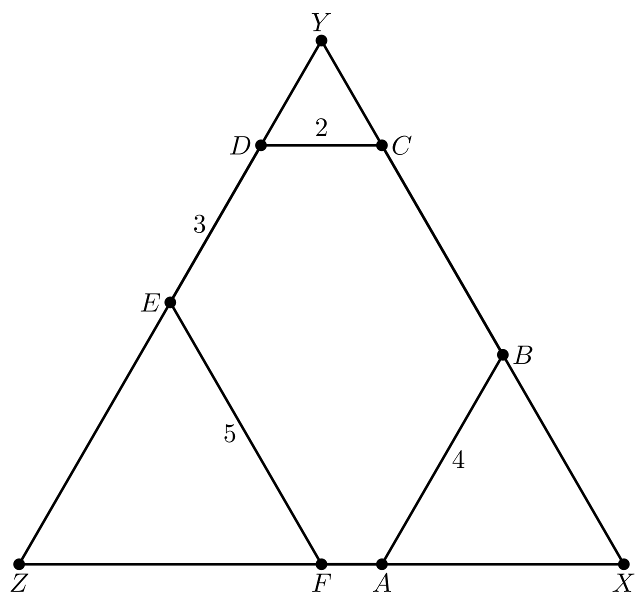
Let hexagon $ABCDEF$ be equiangular. Let $AB = 4$, $CD = 2$, $DE = 3$, and $EF = 5$. Calculate the area of $ABCDEF$.



Solution: Since $ABCDEF$ is equiangular, all of its angles equal

$$\frac{180 \cdot (6-2)}{6}^{\circ} = 120^{\circ}$$

Extend lines AF, BC, and DE so that they intersect at X, Y, and Z, as shown below.



Notice that the supplementary angles formed as a result all equal to 60°. Therefore, $\triangle ABX$, $\triangle CDY$, and $\triangle EFZ$ are all equilateral triangles. Additionally, since $\angle AXB = \angle CYD = \angle EZF = 60^{\circ}$, $\triangle XYZ$ is equilateral. The side length of $\triangle XYZ$ is

$$YZ = DY + DE + EZ = DC + DE + EF = 2 + 3 + 5 = 10$$

Let $[XYZ]$ denote the area of $\triangle XYZ$. We have

$$[ABCDEF] = [XYZ] - [ABX] - [CDY] - [EFZ] = \frac{10^2\sqrt{3}}{4} - \frac{2^2\sqrt{3}}{4} - \frac{4^2\sqrt{3}}{4} - \frac{5^2\sqrt{3}}{4} = \boxed{\frac{55\sqrt{3}}{4}}$$

□

Problem 9

Brian has a gambling addiction. Every day, he spends 2 dollars to buy a lottery ticket. On any given day, there is a 15 dollar prize that can be won with a probability of $\frac{1}{10}$. Brian continues until he wins the prize. On average, how much money will he lose?

Solution:

Solution 1: Consider the following fact:

Corollary 1. *Suppose there is a probability of p that Brian wins. Then, it will take, on average,* $\frac{1}{p}$ *days to win once.*

Then, on average, it will take Brian 10 days to win. That's 20 dollars spent total, and 15 dollars won, resulting in $\boxed{5}$ dollars lost.

Solution 2: Suppose that, on average, Brian will spend x dollars to finally win. Consider day one. Firstly, he spends 2 dollars. Then, either he wins with a probability of $\frac{1}{10}$, or he loses with a probability of $\frac{9}{10}$ and is forced to continue. In the case that he is forced to continue, on day 2, he will also, on average, spend x dollars to finally win. So,

$$x = 2 + \frac{9}{10}x \implies \frac{1}{10}x = 2 \implies x = 20$$

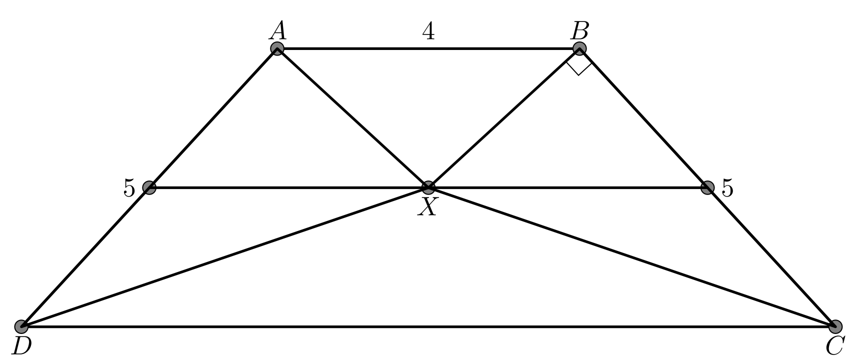
Note that you can also think of this like

$$x = \frac{1}{10} \cdot 2 + \frac{9}{10} \cdot (x + 2)$$

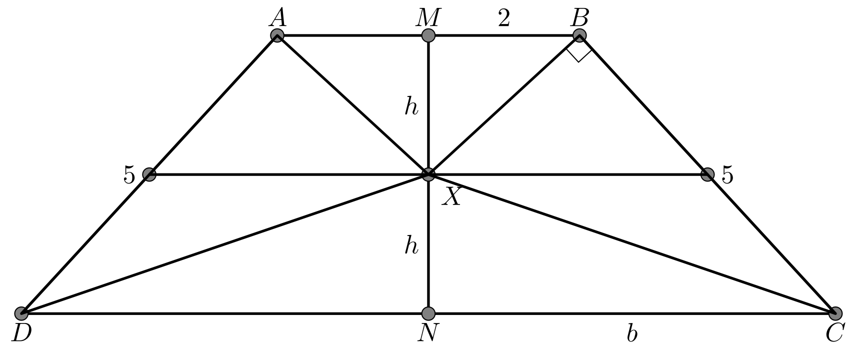
Anyways, he will spend, on average 20 dollars to win, and he will gain 15 dollars. Therefore, he loses, on average $\boxed{5}$ dollars. □

Problem 10

The isosceles trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$, with $AB < CD$. Suppose $AB = 4$ and $BC = DA = 5$. Let the midpoints of BC and DA be E and F, respectively. Let the midpoint of EF (the midline) be X. It is given that $\angle XBC = 90°$. What is the length of $\overline{CD}$?



Solution:



Let M and N be the foots of the perpendiculars from X to $\overline{AB}$ and $\overline{CD}$, respectively. It is clear that they are the midpoints of $\overline{AB}$ and $\overline{CD}$. Since X lies on the midline, let $MX = NX = h$. Also, let $NC = b$ Then,

$$BX^2 = h^2 + (MB)^2 = h^2 + 4$$

and

$$CX^2 = h^2 + b^2$$

Finally,

$$BX^2 + 25 = CX^2 \implies h^2 + 29 = h^2 + b^2 \implies b = \sqrt{29}$$

Thus,

$$CD = 2b = \boxed{2\sqrt{29}}$$

□

Problem 11

What is the smallest positive integer that has exactly 18 positive divisors?

Solution: We know that the number of divisors of n given its prime factorization $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ is

$$(e_1 + 1)(e_2 + 1) \dots (e_k + 1).$$

To make n as small as possible, we want smaller exponents. Therefore, we set $e_1 = 2$, $e_2 = 2$, and $e_3 = 1$, which would force the number of divisors of n to be

$$(2 + 1)(2 + 1)(1 + 1) = 18.$$

Then, we want the larger exponents to be on the smaller primes to minimize n, so we set

$$n = 2^2 \cdot 3^2 \cdot 5 = \boxed{180}.$$

□

Problem 12

There are 8 chairs in a row. There are 4 boys and 3 girls. Two of the boys must sit next to each other and refuse to sit next to girls. On the other hand, the other two boys, Arjun and Andrew, think differently. Arjun does not care whether he sits next to a girl or a boy while Andrew will only sit with two girls immediately to his left and right. How many ways can the 7 people sit down in the 8 chairs (leaving 1 chair empty)? The boys and girls are distinguishable.

Solution: Let BB refer to the first two boys, X be Andrew, and Y be Arjun. Let G be any girl. Let's assume for now that the boys and girls are indistinguishable, and then arrange them as our final step. There are 4 cases:

Case 1: We have

$$\boxed{\text{B}}\boxed{\text{B}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}$$

or

$$\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\text{B}}\boxed{\text{B}}$$

Let's just consider the first one, since they are symmetric. We know that GXG must be together, so we can either have:

1. $\boxed{\text{B}}\boxed{\text{B}}\boxed{\phantom{\text{B}}}\boxed{\text{G}}\boxed{\text{X}}\boxed{\text{G}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}$

   The third G can either go in the 7th or 8th seats (2 options). Y can go either in the 3rd seat, or in the 7/8th seat (whatever one is remaining after the third G is placed, so 2 options). This is a total of $2 \cdot 2 = 4$ arrangements.

2. $\boxed{\text{B}}\boxed{\text{B}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\text{G}}\boxed{\text{X}}\boxed{\text{G}}\boxed{\phantom{\text{B}}}$

   This sub-case is effectively the same as the previous one. 4 arrangements total.

3. $\boxed{\text{B}}\boxed{\text{B}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\text{G}}\boxed{\text{X}}\boxed{\text{G}}$

   This sub-case is effectively the same as the first one. 4 arrangements total.

So, this case has $(4 + 4 + 4) \cdot 2 = 24$ arrangements total.

Case 2: We have

$$\boxed{\phantom{\text{B}}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}$$

or

$$\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\phantom{\text{B}}}$$

Consider just the first one. We can have

1. $\boxed{\phantom{\text{B}}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\phantom{\text{B}}}\boxed{\text{G}}\boxed{\text{X}}\boxed{\text{G}}\boxed{\phantom{\text{B}}}$

   The 3rd girl must go in the 8th seat, while Y can go either in the 1st or 4th seats. 2 arrangements total.

2. $\boxed{\phantom{\text{B}}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\text{G}}\boxed{\text{X}}\boxed{\text{G}}$

   The 3rd girl must go in the 5th seat, while Y can go either in the 1st or 4th seats. 2 arrangements total.

So, this case has $(2 + 2) \cdot 2 = 8$ arrangements total.

Case 3: We have

$$\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}$$

or

$$\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}\boxed{\text{B}}\boxed{\text{B}}\boxed{\phantom{\text{B}}}\boxed{\phantom{\text{B}}}$$

Again, consider just the first one. We must have GXG at the end, and a G at the start. We get

G		B	B		G	X	G

Y must then either be in the 2nd or 5th seats. We get $2 \cdot 2 = 4$ arrangements total.

Case 4: We have

			B	B			

There are no possible arrangements, since GXG cannot be placed anywhere.

In total, we get $24 + 8 + 4 = 36$ arrangements. Now, we multiply by 2! and 3! to arrange the 2 boys and 3 girls. We obtain $36 \cdot 2! \cdot 3! = \boxed{432}$ seating arragements.

□

Problem 13

Let $f(x) = \dfrac{x}{\sqrt{1+2x^2}}$. Let the function $f^{(1)}(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for $n \geq 2$. The value of $f^{(39)}(\frac{1}{\sqrt{3}})$ can be written as the fraction $\dfrac{p}{q}$, where p and q are relatively prime. Find $p+q$.

Solution:

Solution: We begin by computing the composition:

$$f(f(x)) = \frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+2\left(\frac{x}{\sqrt{1+2x^2}}\right)^2}}.$$

Simplifying the denominator gives

$$f(f(x)) = \frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{2x^2}{1+2x^2}}} = \frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{\frac{1+4x^2}{1+2x^2}}} = \frac{x}{\sqrt{1+4x^2}}.$$

This suggests the pattern

$$f^{(n)}(x) = \frac{x}{\sqrt{1+2nx^2}}.$$

Therefore,

$$f^{(39)}\left(\frac{1}{\sqrt{3}}\right) = \frac{\frac{1}{\sqrt{3}}}{\sqrt{1+2(39)\left(\frac{1}{3}\right)}} = \frac{\frac{1}{\sqrt{3}}}{\sqrt{1+26}} = \frac{1/\sqrt{3}}{\sqrt{27}} = \frac{1}{9}.$$

Thus, $p = 1$ and $q = 9$, so

$$p + q = \boxed{10}.$$

□

Problem 14

Mia has 5 distinguishable coins that she wishes to stack on top of one another. Each face has a star on one side but nothing on the other side. How many ways can she stack the coins so that no two adjacent coins have stars that are face to face?

Solution: First, pretend the coins are indistinguishable. Let's represent the tower as a sequence of 5 letters, where the bottom coin is the leftmost letter in the sequence. Let D indicate that the coin's star is facing down and let U indicate that the coin's star is facing up. The pattern we must avoid is "UD", which would create two face to face stars. Therefore, we have the following 6 permissible orientation patterns:

$$DDDDD,\ DDDDU,\ DDDUU,\ DDUUU,\ DUUUU,\ UUUUU.$$

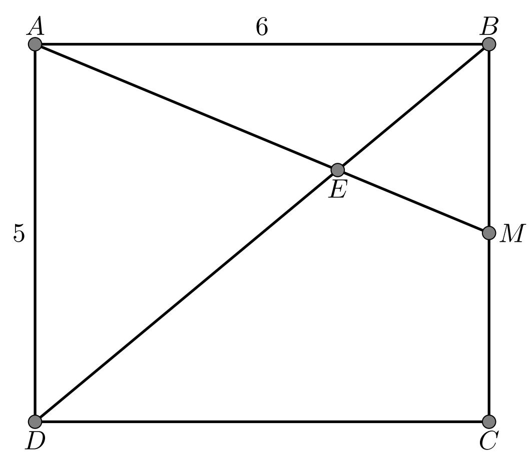
Now, because the coins are distinguishable, they can be ordered in $5! = 120$ ways for each of the 6 patterns. Thus, the number of valid arrangements is

$$120 \cdot 6 = \boxed{720}.$$

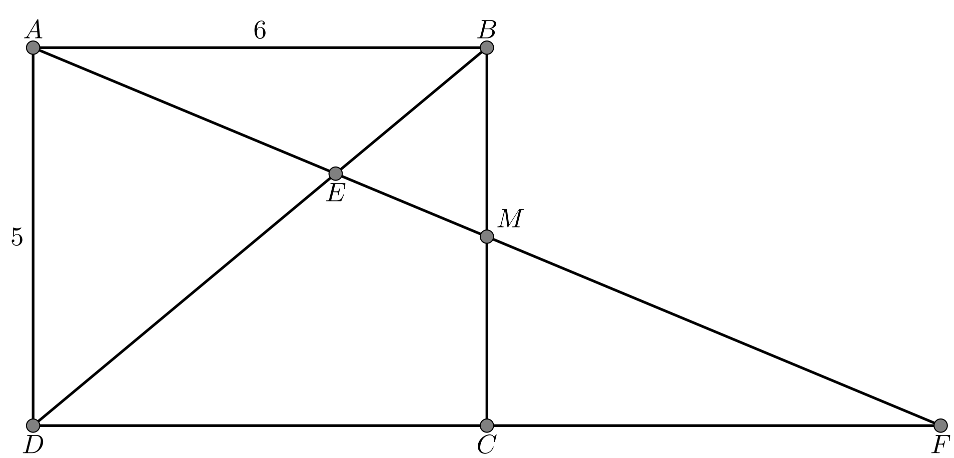
□

Problem 15

Let rectangle $ABCD$ have side lengths $AB = 6$ and $AD = 5$. Let M be the midpoint of $\overline{BC}$. Let the intersection of $\overline{AM}$ and $\overline{BD}$ be E. Find the length of $\overline{AE}$.



Solution:



Extend $\overline{AM}$ to intersect $\overline{CD}$. Let the intersection point be F. Since $\triangle ABM \cong \triangle FCM$ by by ASA congruence, $CF = AB = 6$. So, $FD = 6 + 6 = 12$. Since $\triangle AEB \sim \triangle FED$ by AA similarity,

$$\frac{AE}{EF} = \frac{AB}{FD} = \frac{1}{2} \implies \frac{AE}{AF} = \frac{1}{3}$$

By the Pythagorean theorem,

$$AF = \sqrt{AD^2 + FD^2} = \sqrt{5^2 + 12^2} = 13$$

Therefore,

$$AE = \frac{AF}{3} = \boxed{\frac{13}{3}}$$

□

Problem 16

Ronny is writing positive integers on a board, starting from 1 and counting up consecutively. However, he refuses to write the digits 6 or 7, so he skips any number containing them. For example, after writing 159, the next number he writes is 180. When Ronny writes the number 1013, how many numbers has he actually written on the board?

Solution: Solution 1: We can treat this as a base 8 problem by defining the function f that maps the base 10 digits to the base 8 digits as follows:

$$f(0) \to 0_8,\ f(1) \to 1_8, f(2) \to 2_8,\ f(3) \to 3_8,\ f(4) \to 4_8,\ f(5) \to 5_8,\ f(8) \to 6_8,\ f(9) \to 7_8.$$

Each base 8 number corresponds to exactly 1 valid base 10 number that avoids 6 and 7. For example, $f^{-1}(167_8) = 189$ is a valid number. Using this, we realize that by converting the number Ronny writes on the board from base 8 to base 10, we get the amount of numbers Ronny has written on the board at that point. In particular,

$$1013_8 = 1 \cdot 8^3 + 1 \cdot 8 + 3 = 512 + 8 + 3 = \boxed{523}.$$

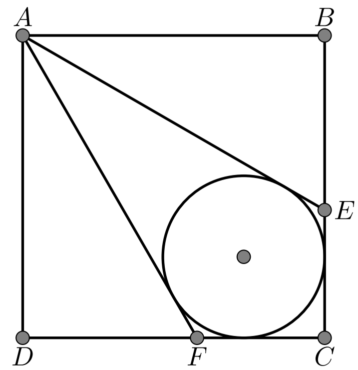
Solution 2: We do casework on the number of digits in the number. For one digit numbers, there are $9 - 2 = 7$ numbers avoiding 6 and 7. For 2 digit numbers, the first digit has 7 choices, while the second digit has 8 choices (including 0). For the 3 digit numbers, the first digit has 7 chcoies while the second and third digit have 8 choices each. Finally, there are 12 four digit numbers between 1000 and 1013 inclusive that avoid 6 and 7. Thus, the total is

$$7 + 7 \cdot 8 + 7 \cdot 8 \cdot 8 + 12 = 7 + 56 + 448 + 12 = \boxed{523}.$$

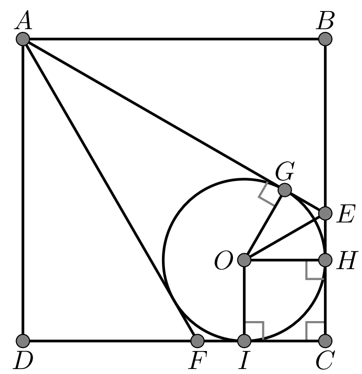
□

Problem 17

Let $ABCD$ be a square with side length 3. Define points E and F on sides BC and CD, respectively, such that $\angle BAD$ is trisected by lines $\overline{AE}$ and $\overline{AF}$. Let circle Ω be tangent to $\overline{AE}$, $\overline{AF}$, $\overline{BC}$, and $\overline{CD}$. Find the radius of circle Ω.



Solution: Let the point of tangency to $\overline{AE}$, $\overline{EC}$ and $\overline{CF}$ be G, H, and I, respectively.



Since $\angle BAD$ is trisected, $\angle BAE = \frac{90^\circ}{3} = 30^\circ$. Thus, $\angle AEC = \angle BAE + \angle ABE = 30^\circ + 90^\circ = 120^\circ$. Since $\triangle OGE \cong \triangle OHE$ by HL congruence, $\angle OEG = \angle OEH$. Since $\angle OEG + \angle OEH = \angle AEH = 120^\circ$, we have $\angle OEH = \frac{120^\circ}{2} = 60^\circ$.

Thus, $\triangle OEH$ is a 30°-60°-90° triangle. Let the radius of the circle be r. Then, $EH = \frac{OE}{\sqrt{3}} = \frac{r\sqrt{3}}{3}$. Additionally, since $\angle OHC = \angle HCI = \angle CIO = 90^\circ$, $OHIC$ is a rectangle. Additionally, $OH = OI = r$, so it is a square. Thus, $HC = OH = r$. Now, we obtain

$$EC = EH + HC = \frac{r\sqrt{3}}{3} + r$$

Since $\triangle ABE$ is also a 30°-60°-90° triangle, we have

$$EC = 3 - BE = 3 - \frac{AB}{\sqrt{3}} = 3 - \frac{3}{\sqrt{3}} = 3 - \sqrt{3}$$

So,

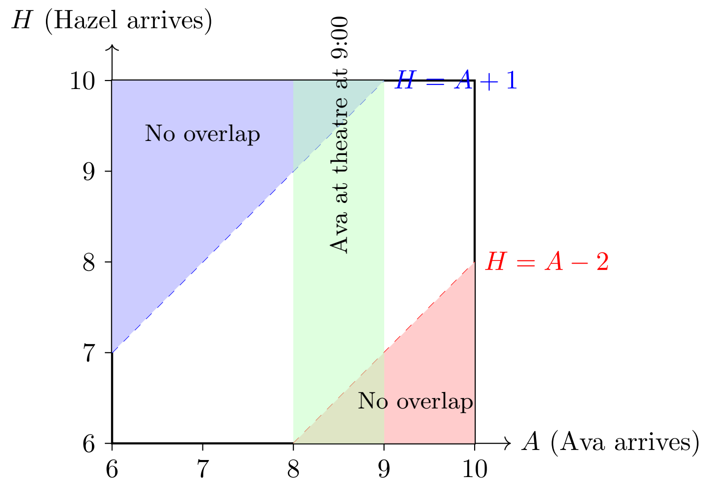
$$\frac{r\sqrt{3}}{3} + r = 3 - \sqrt{3} \implies r(3+\sqrt{3}) = 9 - 3\sqrt{3} \implies r = \frac{9-3\sqrt{3}}{3+\sqrt{3}} = \boxed{6 - 3\sqrt{3}}$$

□

Problem 18

Ava and Hazel go to Proctors Theatre on the same day. Each shows up separately at a random time between 6:00 pm and 10:00 pm. Ava stays for 1 hour, whereas Hazel stays for 2 hours. Given that there does not exist a team where Ava and Hazel where at Proctors Theatre, what is the probability that Ava was at the Theatre at 9:00 pm?

Solution: Sketch the following diagram.



The total overlap area (blue + red) is $\frac{9}{2} + 2 = \frac{13}{2}$. Meanwhile, the condition that Ava is at the Theatre at 9:00 is equivalent to the condition that Ava arrives between 8:00 and 9:00. Computing the overlap area between $B = 8$ and $B = 9$, we get 1. Therefore, the conditional probability is

$$\frac{1}{\frac{13}{2}} = \boxed{\frac{2}{13}}.$$

□

Problem 19

Let x, y and z be positive, real numbers that sum to 2. Find the maximum value of x^4y^3z.

Solution: We can rewrite x^4y^3z as the product of 8 terms, $x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot z$. Using the AM-GM (arithmetic mean-geometric mean) inequality, we know that the arithmetic mean of the sequence must be greater than or equal to the geometric mean. x^4y^3z is the geometric mean of that series.

To get the arithmetic mean in the form of our constraint, we can replace x with $\dfrac{x}{4}$ and y with $\dfrac{y}{3}$, getting

$$\frac{x}{4} + \frac{x}{4} + \frac{x}{4} + \frac{x}{4} + \frac{y}{3} + \frac{y}{3} + \frac{y}{3} + z \geq 8\sqrt[8]{\frac{x^4y^3z}{4^4 3^3}}$$

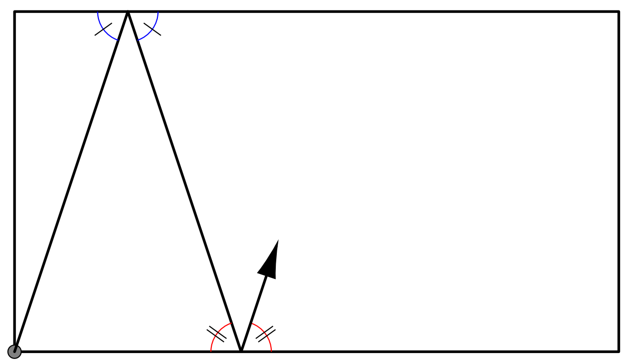
Note that there is an 8 on the right side because the arithmetic mean finds the average of the 8 terms, so to remove it from the denominator of the left side, we multiply both sides by 8.

The left side simplifies to $x + y + z = 2$, so we have $2 \geq \sqrt[8]{\dfrac{x^4y^3z}{4^4 3^3}}$ Raising both sides to the 8th power yields

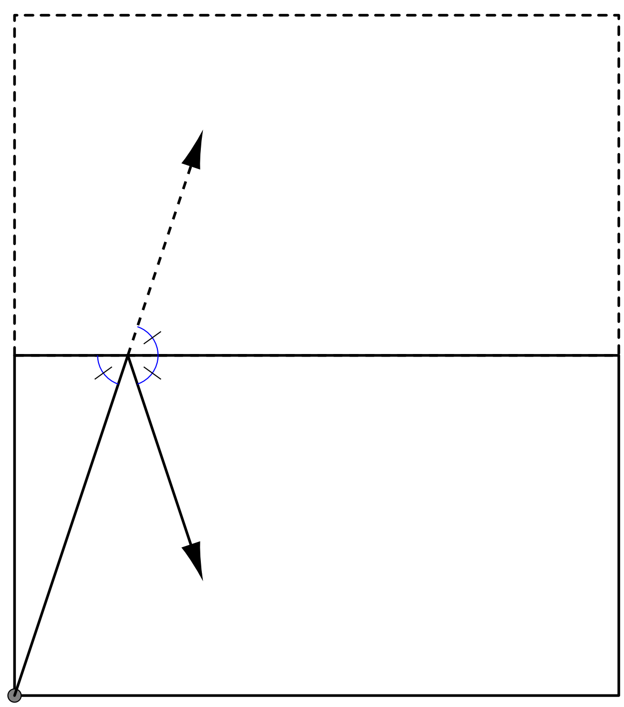
$2^8 \geq 8^8 \cdot \left(\frac{x^4y^3z}{4^4 3^3}\right)$. We can rewrite 8^8 as 2^{24} and 4^4 as 2^8 to yield $\frac{2^8 \cdot 2^8 \cdot 3^3}{2^{24}} \geq x^4y^3z$, and simplifying yields $\frac{27}{256} \geq x^4y^3z$. our answer for the maximum value is this $\frac{27}{256}$ $\square$

Problem 20

A screensaver is played on a 9-inch by 16-inch rectangular computer screen (where the 9-inch sides are vertical). An infinitely small ball begins at the bottom right corner of the screen. Initially, the point travels diagonally with a slope of 3, moving at a constant speed of one inch per second. Every time the point hits a side, it bounces off, leaving at the same angle it approached with. How long will it take for the point to reach another corner?



Solution: Every time the ball hits an edge, rather than bouncing off, allow it to continue to move diagonally. This is equivalent to reflecting the screen across that edge. See below.



Notice that since allowing the line to continue diagonally is equivalent to reflecting the entire screen, it does not change when our ball hits a corner, since the screen is simply just being reflected, and the ball is still in the same location relative to the corners of the newly reflected screen.

So, we form an infinite grid of screens. This reduces the problem down to finding when the diagonal path first contains a point of the form $(16m, 9n)$ for some positive integers m and n. All points on the diagonal path are of the form $(x, 3x)$. So, we set

$$x = 16m, 3x = 9n \implies 16m = 3n$$

So, 16 divides n and 3 divides m. The smallest positive integer (m, n) satisfying these conditions is $(3, 16)$. Thus, $x = 16m = 48$. Our final point will then be $(x, 3x) = (48, 144)$. The total length of the path is

$$\sqrt{48^2 + 144^2} = 48\sqrt{1^2 + 3^2} = 48\sqrt{10}$$

Therefore, the total time it will take is $\boxed{48\sqrt{10}}$ seconds. □

Problem 21

How many of the first 20 problems in this team round do you think your team got right?

Solution: Depends. □

Problem 22

What is the value, in dollars, of a stack of pennies that reaches the moon?

Solution: $2.53 billion. □

Problem 23

How many characters are in the VAMO 2026 Team Round Problems and Solutions file?

Solution: 35621

□