VAMO 2026 Division A Problems and Solutions

Written by the VAMO Team

Problem 1

A passcode for a certain lock is a 3-digit sequence of the integers 1,2, and 3, such as 132. How many passcodes contain the digit 2 at least once?

Solution: The number of sequences is $3^3 = 27$. The number of passcodes that do not contain the digit 2 is $2^3 = 8$, hence the number we seek is $27 - 8 = \boxed{19}$. $\square$

Problem 2

For the function $f(x)$, $f(2) = -6$ and $f(4) = 14$. If $f(x)$ can be written in the form x^2+ax+b, find $a + b$.

Solution: Since $f(2) = -6$, $2^2 + 2a + b = -6$. This simplifies to $2a + b = -10$. Similarly, since $f(4) = 14$, $16 + 4a + b = 14$, so $4a + b = -2$. We can subtract the two equations to get $2a = 8$, so $a = 4$. If $a = 4$, then $8 + b = -10$ so $b = -18$. $a + b = 4 + (-18) = \boxed{-14}$. $\square$

Problem 3

How many positive integers divide both 240 and 336?

Solution: We have $\gcd(240, 330) = 48$, which factorizes as $48 = 2^4 \cdot 3$ and thus has $(4 + 1)(1 + 1) = \boxed{10}$ divisors. $\square$

Problem 4

What is the largest integer n such that 3^n divides $(6!)^4$. Recall that $6! = 6 \times 5 \times \cdots \times 1$.

Solution: The largest power of 3 that divides $6!$ is 3^2. Thus, the largest power of 3 that divides $(6!)^4$ is $4 \cdot 2 = \boxed{8}$. $\square$

Problem 5

Two diagonals of a regular hexagon are chosen at random. What is the probability that they intersect at a point in the interior of the hexagon?

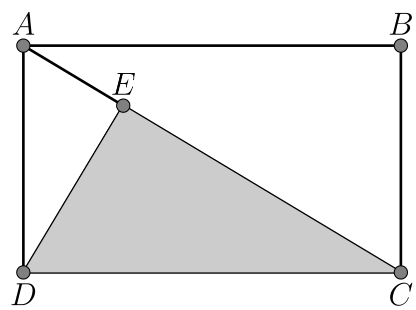
Solution: The number of diagonals in a hexagon is 9 (whether by drawing them out or using the formula $\frac{n(n-3)}{2}$. There are 3 long diagonals that divide the hexagon perfectly in half. These diagonals intersect 4 of the 8 other diagonals, with a probability of $\frac{4}{8} = \frac{1}{2}$. The other 6 shorter diagonals intersect 3 other diagonals, with a probability of $\frac{3}{8}$. Therefore, the overall probability is

$$\frac{3}{9} \cdot \frac{1}{2} + \frac{6}{9} \cdot \frac{3}{8} = \frac{1}{6} + \frac{1}{4} = \boxed{\frac{5}{12}}$$

□

Problem 6

Rice farmer Brian has a rectangular field $ABCD$ of lengths 3 and 4. The land above the diagonal $\overline{AC}$ is unusable. Additionally, Brian marks off part of the field for animal use. To do this, he walks from D to $\overline{AC}$ so that his path is perpendicular to $\overline{AC}$ and ends at E. What is the area of $\triangle CDE$?



Solution: By AA similarity, $\triangle CDE$ and $\triangle CAD$ are similar. $\overline{AC}$ has length $\sqrt{3^2 + 4^2} = 5$ by the Pythagorean theorem. By similarity, we have that

$$\frac{DE}{AD} = \frac{CE}{CD} = \frac{CD}{AC} \implies \frac{DE}{3} = \frac{CE}{4} = \frac{4}{5}$$

Thus, $DE = 3 \cdot \frac{4}{5} = \frac{12}{5}$, and $CE = 4 \cdot \frac{4}{5} = \frac{16}{5}$. Hence, the area of $\triangle CDE$ is

$$\frac{1}{2} \cdot DE \cdot CE = \frac{1}{2} \cdot \frac{12}{5} \cdot \frac{16}{5} = \boxed{\frac{96}{25}}$$

□

Problem 7

Find the smallest positive integer k for which the functions $f(x) = x^2 + k$ and $g(x) = kx$ intersect.

Solution: When functions intersect, their difference is 0. So, we can write $0 = x^2 + k - kx$, which is a standard quadratic function. We can use the quadratic formula to get

$$x = \frac{k \pm \sqrt{k^2 - 4k}}{2}$$

, For x to be a real number, $k^2 - 4k \geq 0$. We can factor the left hand side to get $k(k-4) \geq 0$. This is true when $k \leq 0$ or $k \geq 4$ so the least positive integer k where these functions intersect is $k = 4$. $\square$

Problem 8

What are the last two digits of 7^{2026}?

Solution: We look for a pattern in the powers of 7 mod 100.

$$\begin{aligned}
7^1 &\equiv 7 \pmod{100} \\
7^2 &\equiv 49 \pmod{100} \\
7^3 &\equiv 43 \pmod{100} \\
7^4 &\equiv 1 \pmod{100}.
\end{aligned}$$

Therefore, the last two digits of the power have a cycle of length 4. Because $2026 \equiv 2 \pmod{4}$, the last two digits of 7^{2026} are $\boxed{49}$. $\square$

Problem 9

How many ordered triples of positive integers (a, b, c) satisfy $a + b + c = 26$?

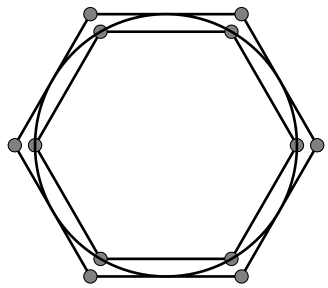
Solution: One can think of this as distributing 26 balls among 3 distinguishable boxes, where each box must have at least one ball. Thus, we do stars and bars on 26-3 = 23 balls and 3 boxes:

$$\binom{23+3-1}{3-1} = \binom{25}{2} = \boxed{300}.$$

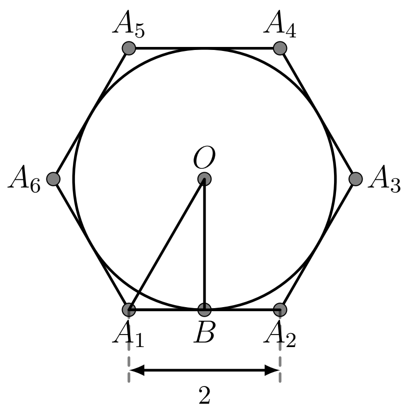
$\square$

Problem 10

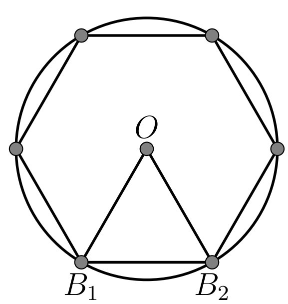
A regular hexagon has a side length of two. It circumscribes a circle Ω centered at O. Circle Ω circumscribes a regular hexagon as well. What is the length of the sides of the interior hexagon?



Solution: Firstly, we find the radius of the circle.



Above, notice that $\angle BA_1O = 60$ and $\angle OBA_1 = 90$. So, OA_1B is a $30-60-90$ triangle. B is the midpoint of $\overline{A_1A_2}$, so $A_1B = 1$. Thus, $OB = \sqrt{3}$.



Now, notice that $\triangle OB_1B_2$ is equilateral, since $\angle OB_1B_2 = \angle OB_2B_1 = 60°$. Therefore, since the radius of the circle is $\sqrt{3}$, we have $OB_1 = OB_2 = B_1B_2 = \boxed{\sqrt{3}}$ (B_1B_2 is the desired side length).

□

Problem 11

Find the positive integer n that can be expressed as $\sqrt[3]{24+\sqrt[3]{24+\sqrt[3]{24+\dots}}}$

Solution: Let

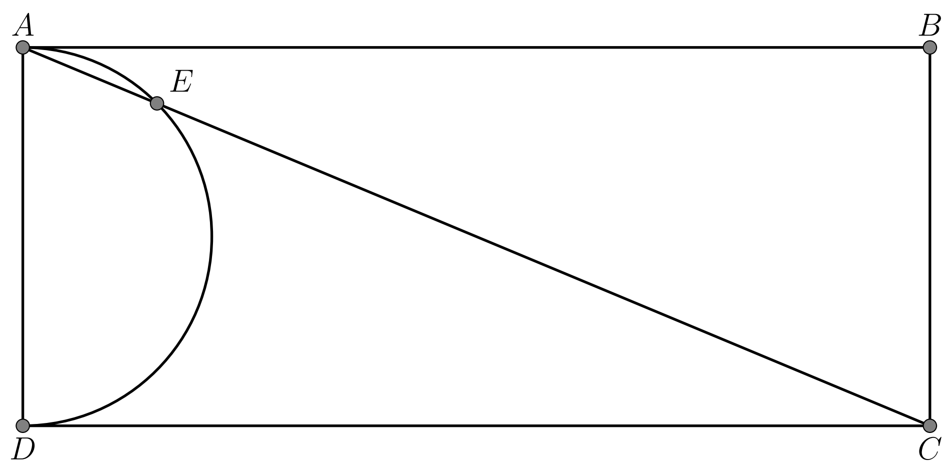
$$n = \sqrt[3]{24+\sqrt[3]{24+\sqrt[3]{24+\dots}}}$$

$$n^3 = 24 + \sqrt[3]{24+\sqrt[3]{24+\sqrt[3]{24+\dots}}}$$

. We can see that the second half of the right side looks a lot like n. Thus, $n^3 = 24 + n$ so $n^3 - n - 24 = 0$. Since we know n must be an integer, we can begin testing integers less than 24. $n = 1$ and $n = 2$ don't work, but $n = 3$ does. Thus, $n = \boxed{3}$ $\square$

Problem 12

Let rectangle $ABCD$ have side lengths 5 and 12, where $AB = 12$. Consider a semicircle Ω with diameter AD. Let the intersection of semicircle Ω and AC be E. What is the length of AE?



Solution: Firstly, by the Pythagorean theorem,

$$AC = \sqrt{AD^2 + CD^2} = \sqrt{5^2 + 12^2} = 13$$

Next, note that $\angle AED = 90°$ by Thales's Theorem. Additionally, $\angle DAE = \angle CAD$, so $\triangle DAE \sim \triangle CAD$. Thus,

$$\frac{AE}{AD} = \frac{AD}{AC} \implies AE = \frac{AD^2}{AC} = \boxed{\frac{25}{13}}$$

Alternatively, one can use Power of a Point, considering the power of C with respect to Ω. This gives us

$$CD^2 = CE \cdot CA \implies CE = \frac{CD^2}{CA} = \frac{144}{13}$$

Thus,

$$AE = CA - CE = 13 - \frac{144}{13} = \boxed{\frac{25}{13}}.$$

□

Problem 13

A rectangular prism has side lengths equal to the roots of the polynomial $2x^3 - 25x^2 + 80x - 50$. The ratio of its surface area to volume can be written as a common fraction $\frac{p}{q}$ where p and q are relatively prime. Find $p+q$.

Solution: Let the roots be a, b, and c. By Vieta's theorems, we know that $ab + bc + ac = \frac{c}{a}$ and $abc = -\frac{d}{a}$. The surface area is $2ab + 2bc + 2ac$ so it's $\frac{2c}{a}$ and the volume is abc which is $\frac{d}{a}$. The ratio is $\frac{2c}{d}$ or $\frac{160}{50} = \frac{16}{5}$. $16 + 5 = \boxed{21}$. □

Problem 14

Andrea and Bob are playing a game where a player wins when they are 2 points ahead of the opponent. If Andrea has a $\frac{5}{8}$ probability of winning each point and Bob has a $\frac{3}{8}$ probability of winning each point, what is the probability that Andrea wins the whole game?

Solution: We have three states: Andrea is ahead by 1, the score is tied, and Bob is ahead by 1. Let p be the probability that Andrea wins starting from the score tied (first at 0–0). In the next 2 points, we have 4 cases: Andrea wins both and wins the whole game, Andrea wins one then Bob wins one, Bob wins one then Andrea wins one, and Bob wins both and wins the whole game. The first case where Andrea wins the next 2 points occurs with a probability of $\frac{5}{8} \cdot \frac{5}{8} = \frac{25}{64}$. The probability that Andrea wins one and then Bob wins one is $\frac{5}{8} \cdot \frac{3}{8} = \frac{15}{64}$, and likewise the probability that Bob wins one and then Andrea wins one is $\frac{3}{8} \cdot \frac{5}{8} = \frac{15}{64}$. In these two cases, we return back to the original state of a tied score, with the probability p

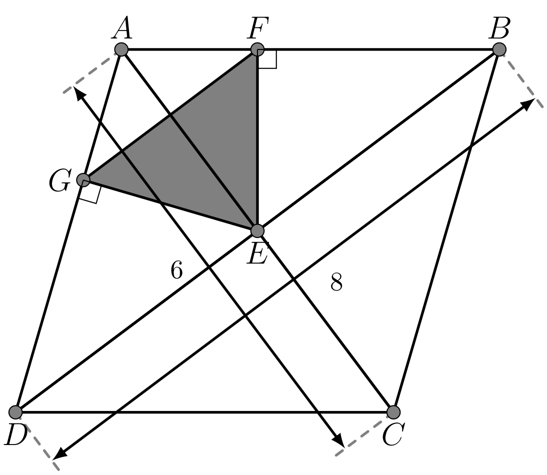
that Andrea wins. Thus,

$$\begin{aligned}
p &= \frac{25}{64} + 2 \cdot \frac{15}{64}p = \frac{25}{64} + \frac{30}{64}p \\
\frac{34}{64}p &= \frac{25}{64} \\
p &= \boxed{\frac{25}{34}}.
\end{aligned}$$

□

Problem 15

Suppose we have a rhombus $ABCD$ whose diagonals AC and BD have lengths 6 and 8, respectively. Let the intersection of the diagonals be E. Let the feet of the perpendiculars from E to the sides of the rhombus be F and G, respectively. Find the area of $\triangle EFG$.



Solution:

Note that E is the midpoint of the diagonals, and that the diagonals are perpendicular. Thus, $AE = \frac{*}{2} = 3$ and $BE = \frac{10}{2} = 4$, so by the Pythagorean theorem,

$$AB = \sqrt{AE^2 + BE^2} = \sqrt{3^2 + 4^2} = 5$$

We use areas now. The area of $\triangle AEB$ is $\frac{1}{2} \cdot 3 \cdot 4 = 6$. The area of $\triangle AEB$ is also $\frac{1}{2} \cdot EF \cdot AB$. Thus,

$$EF \cdot \frac{5}{2} = 6 \implies EF = \frac{12}{5}$$

Likewise, we find that $GE = \frac{12}{5}$.

Now, let GF and AE intersect at H. Notice that H is the midpoint of $\overline{GF}$, and that $\overline{GF} \perp \overline{AE}$. Since

$$\angle HEF = \angle AEF = 90 - \angle FAE = 90 - (90 - \angle ABE) = \angle ABE$$

and since $\angle FHE = \angle AEB = 90^\circ$, $\triangle FHE \sim \triangle AEB$. Therefore,

$$\frac{FH}{FE} = \frac{AE}{AB} \implies FH = \frac{AE}{AB} \cdot FE = \frac{3}{5} \cdot \frac{12}{5} = \frac{36}{25}$$

Additionally,

$$\frac{HE}{FE} = \frac{BE}{AB} \implies HE = \frac{BE}{AB} \cdot FE = \frac{4}{5} \cdot \frac{12}{5} = \frac{48}{25}$$

Thus, the area of $\triangle EFG$ is

$$\frac{1}{2} \cdot GF \cdot HE = \frac{1}{2} \cdot (2 \cdot HF) \cdot HE = HF \cdot HE = \frac{36}{25} \cdot \frac{48}{25} = \boxed{\frac{1728}{625}}$$

$\square$